Descartes Rules Polynomials with Signs

A project done by

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1. Introduction

René Descartes initially stated his Rule of Signs – without any proof, however. Refer to Bartolozzi et al. [1]. Isaac Newton later restated the rule but thought the proof was too trivial to warrant a recording. Although the earliest known proof was by the French mathematician Jean-Paul de Gua de Malves, it was the German mathematician Carl Friedrich Gauss who showed that the difference between the number of sign changes and the number of positive roots of a polynomial is negative. Refer to Hosch [4].

Consider a polynomial $p(x) = p_0x^0 \ldots + p_nx^n$ (that is, with terms arranged by increasing powers of x) with real coefficients. Without further mention, we assume that the highest coefficient p_n is non-zero.

Descartes' Rule of Signs states that the number of positive roots (**counted with multiplicity**) of p does not exceed the number of sign changes of the nonzero coefficients of p when arranged in the order $p_0, p_1, p_2, \ldots, p_n$. More precisely, the rule states that the number of such sign changes minus the number of positive roots is a nonnegative even number.

If $Z_{positive}$ and S represent the number of positive roots of p and the number of sign changes in p respectively, then Descartes' principle can be written concisely:

Theorem [Descartes' Rule of signs] *$S - Z_{positive} = 2k$, where k is a nonnegative integer*.

We again emphasise that zero coefficients are skipped when counting a sign change.

A quick corollary of this result is that a polynomial with an odd number of sign changes has at least one positive root!

We remark that the **Descartes Rule is significant since it relates an easy-to-calculate quantity (coefficient sign changes) of a polynomial with a difficult-to-calculate quantity (the number of real roots)**. See Appendix for a very simple Python program that calculates the possible number of positive, negative, and zero roots of a given polynomial.

In this article, we follow the arguments outlined by Levine [3]. Our addition to Levine's work involves an expansion of and many minor corrections to the points and proofs made in his article. Section 2.1 uses a proof on a simple trinomial to illustrate the rule, while section 2.2 illustrates the intuition behind the rule with the help of graphs. The arguments for the proof will be given in two sections. In section 2.3, we will prove some preliminary results as well as simpler versions of the rule. In section 2.4, we will prove the rule itself. Section 2.5 explains the watertight nature of the rule of signs by showing its versatility.

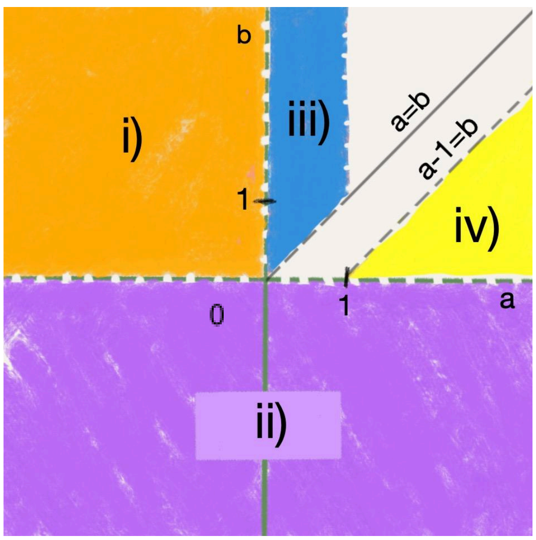
2. Analysis

2.1. Using a Simple Polynomial to Illustrate the Rule

Lemma 1 *For arbitrary powers $n > m > 0$, consider a polynomial of the form $1 - ax^m + bx^n$ with real coefficients a and b. The number of positive roots and coefficient sign changes for the polynomial are given in the following table:*

Coefficient inequalities	*Number of positive roots*	*Number of sign changes*
i) $a < 0$, $b > 0$	*0*	*0*
ii) $b < 0$	*1*	*1*
iii) $\min(b,1) > a > 0$	*0*	*2*
iv) $a - 1 > b > 0$	*2*	*2*

The figure below gives the ranges of a and b being considered.



Proof. For i) all the terms are positive and so the trinomial is positive for all $x > 0$, making a zero crossing impossible there.

For ii) we rewrite the equation

$$1 - ax^m + bx^n = 0$$

as

$$1 - ax^m = -bx^n.$$

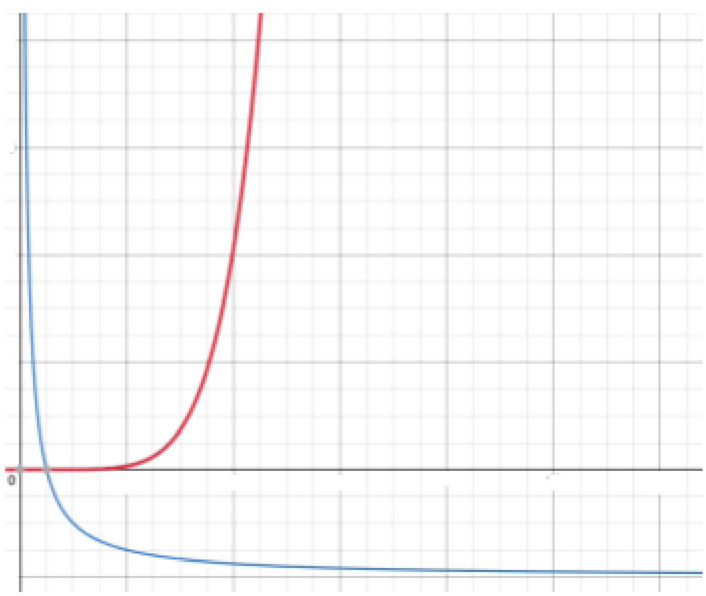
Since $b < 0$, we can rewrite the above equation as

$$1 - ax^m = |b|x^n.$$

Dividing on both sides by x^m, we get

$$x^{-m} - a = |b|x^{n-m}.$$

The meeting point/s, if any, of the two functions $x^{-m} - a$ and $|b|x^{n-m}$ is/are the roots of our original trinomial – if the trinomial has any real roots at all. The right hand side strictly increases and the left hand side strictly decreases in $x > 0$. So, there is at most one positive root. As x nears 0, the right hand side (red curve in the figure below) approaches zero and the left hand side (blue curve in the figure below) approaches ∞. As x tends to ∞, the left hand side approaches $-a$ and the right hand side approaches ∞. Note that $-a$ needn't always be negative.



Therefore, by continuity, the two curves cross in $x > 0$ and there is exactly one root. Recall that a function defined on the real line is deemed continuous if its graph can be drawn on a piece of paper without lifting the pen.

For cases iii) and iv), noting that $a > 0$ and $b > 0$, we rewrite the equation

$$0 = 1 - ax^m + bx^n$$

as

$$ax^m = 1 + bx^n$$

and then divide by ax^m to get

$$1 = \frac{1}{a}x^{-m} + \frac{b}{a}x^{n-m}.$$

Again, the meeting point/s of the two functions $a^{-1}x^{-m} + ba^{-1}x^{n-m}$ and $y = 1$ is/are the roots of our original trinomial, if the trinomial has any real roots at all.

The values of x^{-m} and x^{n-m} in $x > 0$ can be summarized as

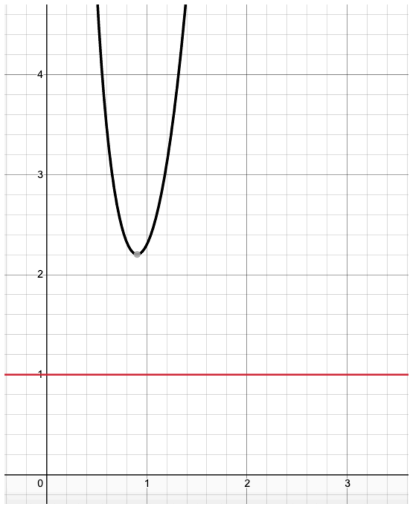
For $0 < x < 1$,	$x^{-m} > 1$	$x^{n-m} < 1$
For $x = 1$,	$x^{-m} = 1$	$x^{n-m} = 1$
For $x > 1$,	$x^{-m} < 1$	$x^{n-m} > 1$

In the case of iii), since a is always between 0 and 1 and b is always greater than a, $\frac{1}{a}$ and $\frac{b}{a}$ are both greater than 1.

For each of the three ranges (or values) of x given in the table above, the right hand side (black curve in the figure below) of

$$1 = \frac{1}{a}x^{-m} + \frac{b}{a}x^{n-m}$$

is greater than 1 (red line in the figure below) in $x > 0$, and so the equality cannot be satisfied for $x > 0$.



For iv) we rewrite

$$a - 1 > b$$

as

$$a > b + 1$$

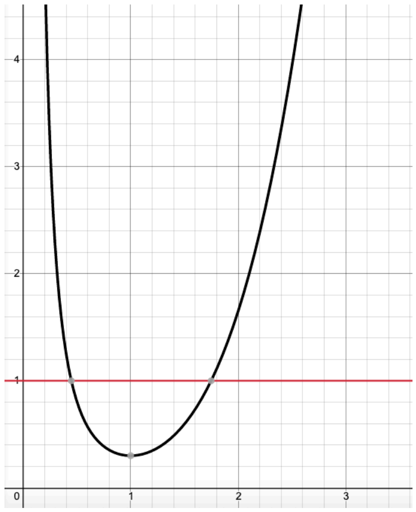
and then divide by *a* to get (recall a is positive)

$$1 > \frac{1}{a} + \frac{b}{a}.$$

So, the right hand side (black curve in the figure below) of

$$1 = \frac{1}{a}x^{-m} + \frac{b}{a}x^{n-m} \qquad \ldots\ a)$$

is less than 1 for $x = 1$ and, as before, greater than 1 for both x approaching 0 and ∞. So, the right hand side and $y = 1$ meet at at least two points. Consequently, the trinomial has at least two positive roots in case iv), one less than 1 and the other greater than 1.



To prove that the trinomial in iv) has exactly two roots, we could show that the right hand side models a convex curve in $x > 0$. In loose terms, a convex curve is one that curves or ‘looks’ upward from the x axis. Showing this helps because a convex curve and a line can intersect at at most two points. Note that we have already shown that there are at least two intersections.

The curve

$$\frac{1}{a}x^{-m} + \frac{b}{a}x^{n-m}$$

has the first derivative

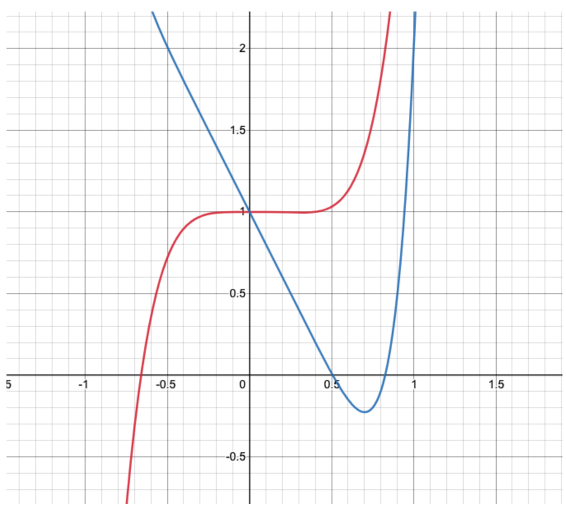
$$-\frac{m}{a}x^{-(m+1)} + \frac{b(n-m)}{a}x^{(n-m)-1}$$

and the second derivative

$$\frac{m(m+1)}{a}x^{-(m+2)} + \frac{b(n-m)(n-m-1)}{a}x^{(n-m-2)}.$$

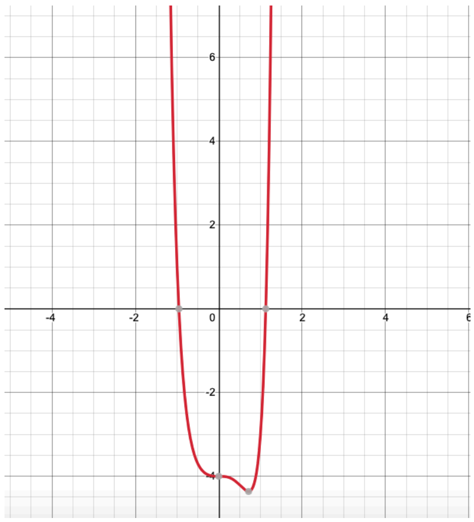
Scrutinizing the second derivative in $x > 0$, since m is positive and lesser than n, the first coefficient is always positive and the second coefficient is either positive or zero; it’s zero in the case where $n - m - 1 = 0$. So, the second derivative is always positive. That is, the slope of the curve is always increasing. An increasing slope signifies a convex function (here, in the domain $x > 0$). A convex curve and a line can intersect at most twice. To prove this by contradiction, let's assume that a third positive root exists. This means that the curve would have to turn back toward the line and intersect it, decreasing its slope in doing so. However, the second derivative of the curve is nonnegative in $x > 0$, so a decreasing slope is not supported, hence our assumption is incorrect. So the equation in *a)* has to have at most two positive roots. Using previous results, we can conclude that the trinomial has exactly two positive roots in case iv).

Note that the trinomial may not have the same number of positive roots as (a,b) varies over the region left unaccounted for by Lemma 1. Let us consider the region (a,b) such that $a > 1$ and $a < b$. This is the unshaded region above the line a = b. The trinomial $q(x) = 1 - 2x^4 + 5x^5$, the red curve in the figure below, with (a,b) in this region, has no positive roots. If instead we consider $p(x) = 1 - 2x + 3x^8$, the blue curve below, it has two positive zeroes. The number of positive roots of a trinomial with (a,b) taken in this region clearly varies with (a,b).



2.2. The Intuition behind Descartes's Rule of Signs

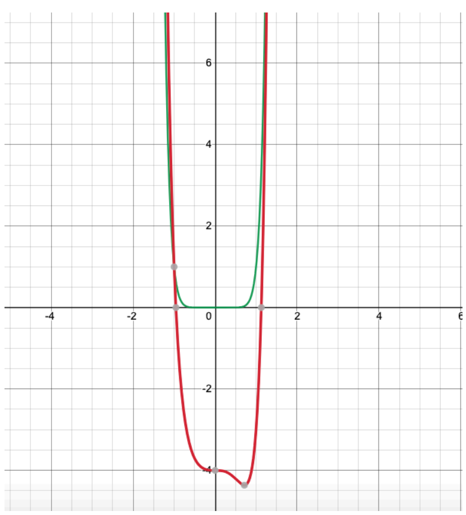
i) Descartes' Rule is plausible when we consider that each power of x dominates in a different region of $x > 0$, as seen below in the case of the polynomial, $p(x) = -4 - 2x^3 + 3x^6 - x^8 + x^{10}$.



We'll only consider *p(x)* in $x > 0$.

ii) When x is very large, then the term with the highest power of x in *p(x)*, say $p_n x^n$, dominates and the sign of *p(x)* is that of the leading coefficient p_n.

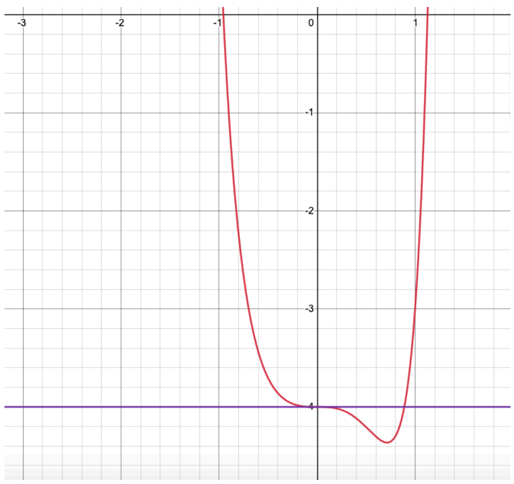
Here, $p_n x^n = x^{10}$ is shown by the green curve in the figure below almost overlapping with *p(x)* (red) for all sufficiently large $x > 0$.



The sign of $p_n = p_{10}$ (positive) is also taken up by *p(x)* in this range of x. Note that red and green never exactly overlap, they only meet when $x = \infty$.

iii) When x is very small, then the term with the lowest power of x, typically p_0, rules.

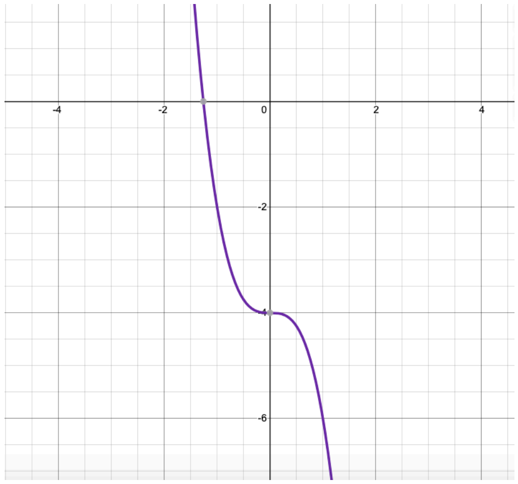
Here, $p_0 = -4$ is shown by the purple line in the figure below almost overlapping with $p(x)$ (red) for all sufficiently small $x > 0$.



The sign of p_0 (negative) is also taken up by $p(x)$ in this range of x. Note again that red and purple never exactly overlap, they only meet when x is 0.

iv) As we move right from the origin, each successive power of x comes into play.

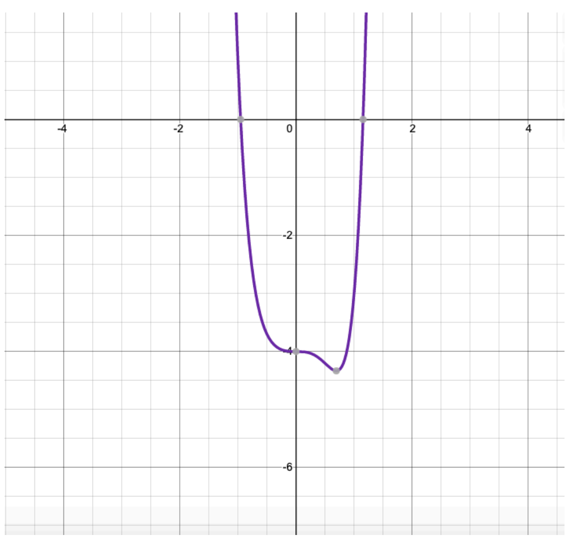
v) If the sign of the coefficient of the next power of x does not change, then the function continues the trend set by the previous power, trending towards negative values if the coefficient is negative or positive values if the coefficient is positive. Let the previous term be – 4 (purple in the figure below). For $p(x)$, the 'new term' is $-2x^3$, and $y = -4 - 2x^3$ is shown by the figure below.



The graph is tending toward negative values in $x > 0$. A similar method can be used to show a graph tending toward positive values. Note that the lead term $-2x^3$ dominates for large $x > 0$, as given by ii).

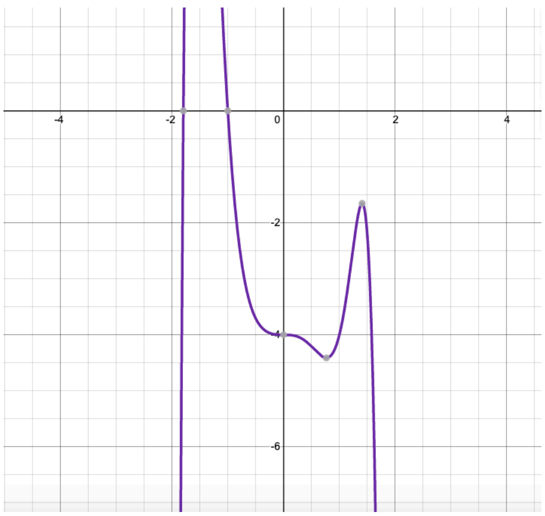
vi) Consequently, if there is to be a zero crossing, then there needs to be a sign change in coefficients.

p(x) next adds $+ 3x^6$, creating a sign change. The polynomial
$y = -4 - 2x^3 + 3x^6$ is shown in the figure below.



Clearly, we now have a zero crossing. Here too, the lead term $+ 3x^6$ dominates for large $x > 0$.

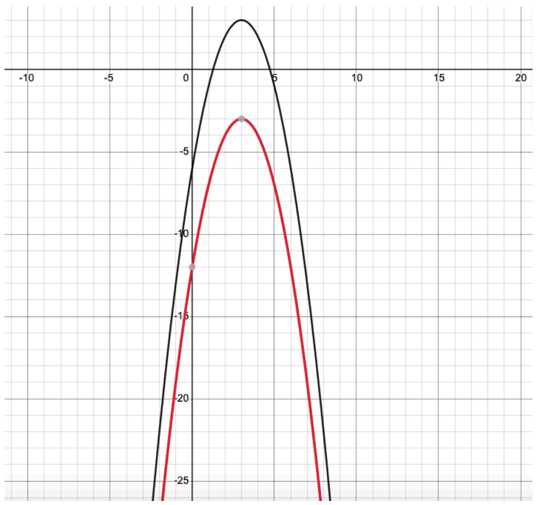
vii) If there is a sign change but there isn't still a zero crossing, as shown by $y = -4 - 2x^3 + 3x^6 - x^8$ in the figure below, then we must have turned away from the x axis due to another sign change (here, due to the $- x^8$ cancelling out the sign change induced by $+ 3x^6$, creating a U-turn away from the positive x axis). Here as well, the lead term $- x^8$ dominates for large $x > 0$.



We will now need to switch signs again to head back towards the x axis, so introducing $+ x^{10}$ into y, giving $y = -4 - 2x^3 + 3x^6 - x^8 + x^{10}$, which is $p(x)$ as shown previously in i).

In going from $y = -4$ to $p(x) = -4 - 2x^3 + 3x^6 - x^8 + x^{10}$, by successively adding each higher power, the leading term (-4, $-2x^3$, $+3x^6$, $-x^8$, and $+x^{10}$ respectively) of each stage can be thought of as dominating the magnitude and sign of its corresponding stage for large $x > 0$, as in point ii). For such a domination to occur, the polynomial will have to reverse the trend of its previous path if the signs of the previously dominant term, say, $+ 3x^6$, and the newly dominant term, $- x^8$, don't agree. That is, each sign change produces a path change in the graph.

Now, this turning/path change can occur before the graph has hit the $+x$ axis as in vii) or as in $f(x) = -12 + 6x - x^2$ (red curve in the figure below). Else, the turn can occur after the graph has hit the x axis as in $q(x) = -6 + 6x - x^2$ (black curve in the figure below). Both polynomials have two sign changes but only the latter results in two positive roots, while the former has zero positive roots, further illustrating why sign changes must be dropped in pairs.



In short, a pair of consecutive sign changes either causes two positive roots or none at all.

2.3. Preliminaries

Remark 1 *We may take the leading coefficient* p_n *of the polynomial* p *to be unity without loss of generality.*

Multiplying or dividing p by any nonzero real number affects neither the location and number of sign changes in its coefficients nor the location and number of its roots. We will continue to employ the symbol p_n when it helps simplify the notation.

Remark 2 *We can safely assume the constant term* p_0 *is nonzero, i.e. the polynomial has no zero roots.*

Removing any common factors of x does not change the number of positive (or negative) roots, but just reduces the number of zero roots, which we are not interested in.

Points ii) and iii) from **section 2.2** (Intuition of the rule) are proved below.

Result 1 *For all sufficiently small $x > 0$, the sign of a polynomial $p(x)$ matches the sign of its trailing coefficient.*

Proof. We write

$$p(x) = p_0 + p_1 x + \ldots + p_n x^n.$$

$p(x) = p_0$ at $x = 0$ and the sign of p_0 dominates $p(x)$. For small $x > 0$, $p(x)$ agrees in sign with p_0. Hence, the trailing coefficient of p, p_0, dominates in this range of sufficiently small $x > 0$. Precisely, the range

$$\text{smallest positive real root of } p = s > x > 0$$

is the range of all sufficiently small $x > 0$ for which the sign of the trailing term determines the sign of p.

Result 2 *For all sufficiently large $x > 0$, the sign of p matches the sign of its leading coefficient.*

Proof. Factoring out $p_n x^n$,

$$p(x) = p_n x^n \left(\left\{ \frac{p_0}{p_n x^n} + \frac{p_1}{p_n x^{n-1}} + \frac{p_2}{p_n x^{n-2}} + \ldots + \frac{p_{n-1}}{p_n x} \right\} + 1 \right)$$

To retain the sign of p_n, the terms within the parentheses must sum to a positive value for all sufficiently large $x > 0$. Let's prove that they do so.

The terms in the parentheses of $p(x)$ can be written as

$$(\{ m \} + 1),$$

where the content in the curly-brackets is abbreviated as m.

When $x = \infty$, $\{ m \}$ is 0 and hence, $(\{ m \} + 1)$ equals 1. As x decreases from ∞ in $x > 0$, $p(x)$ maintains the sign of p_n until $p(x)$ touches (if at all) the x axis for the first time (i.e., where $p(s) = 0$ for some positive s). The range

largest positive real root of $p = s < x < \infty$

is hence the range of all sufficiently large $x > 0$ for which the sign of the leading term determines the sign of p.

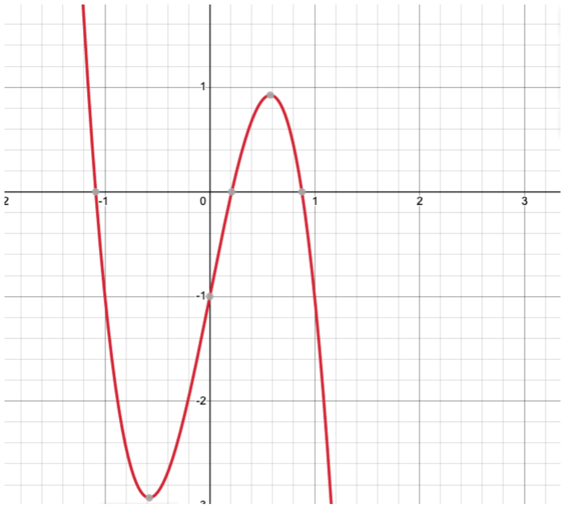
Proposition 1 *If all the nonzero coefficients of p agree in sign, then it has no positive roots.*

Proof. If all the non-zero coefficients are positive, p is a sum of positive terms for any $x > 0$ and so cannot equal zero there. A similar argument holds if all the non-zero coefficients are negative.

Corollary 1 *If all the coefficients of p are nonzero and alternate in sign, then p has no negative roots.*

Proof. $p(-x)$ has all coefficients agreeing in sign, hence Proposition 1 applies. So, $p(-x)$ has no positive roots. That is, p has no negative roots.

Note: For this proof, it's important that all coefficients in $p(x)$ are nonzero. For example, the proof doesn't hold for $p(x) = -1 + 5x - 5x^3$ (figure below), although the coefficients have alternating signs.



Corollary 2 *If a polynomial p with an even number of consecutive zero coefficients has alternating signs in the nonzero coefficients, then it has no negative roots.*

Proof. The two nonzero coefficients on either side of a zero coefficient block have powers of x that are of differing parity due to the fact that this zero coefficient block has an even number of elements. Hence, $p(-x)$ has all coefficients agreeing in sign, hence Proposition 1 applies and we can conclude as in corollary 1.

The next proposition uses proof by induction. Mathematical induction involves showing the validity of an assertion for any natural number (here, the statement of the proposition) via a three-step proof. Mathematical induction can be executed in the following general format:

Hypothesis: Statement $P(n)$ holds where n is any natural number (here, the set of natural numbers includes zero).
Steps of Proof.

1) Base case: Show that $P(0)$ holds
2) Inductive step: Show that $P(k + 1)$ holds given that $P(k)$ holds
3) Conclusion: Since both the base case and the inductive step have been proven, by mathematical induction the statement $P(n)$ holds for every natural number n.

Proposition 2 *If a polynomial p of degree n has n positive roots, then its coefficients are all nonzero and the signs of the coefficients of p alternate.*

Proof. We proceed by induction on n.

1) Base case: We can have two base cases. First, for $n = 0$ there are no roots and no sign changes. Second, for $n = 1$, we write $p(x)$ as $ax + b$. Since p has one positive root, $p(x) = ax + b = 0$ and $x = -b \div a$. Since x is positive, b and a must both be non-zero and differ in sign.

2) Inductive step: Assuming that the proposition holds for any polynomial of degree $n - 1$ having $n - 1$ positive roots, we show that it's true for a polynomial of degree n having n positive roots.

   Proof. Thanks to the inductive step, a polynomial of degree $n - 1$ having $n - 1$ positive roots can be written as (with all p_j's positive)

$$\sum_{j=0}^{n-1} (-1)^{n-1-j} p_j x^j$$

   Therefore, a polynomial p of degree n having n positive roots may be written as the product

$$(x - \alpha) \sum_{j=0}^{n-1} (-1)^{n-1-j} p_j x^j$$

   with α and all p_j positive. Here, α is the n^{th} positive root of the polynomial p.

   Taking $j = 0, 1, \ldots, n - 2, n - 1$ and writing out the summation term, we get

$$p(x) = (x - \alpha)\Big(\big[(-1)^{n-1} p_0\big] + \big[(-1)^{n-2} p_1 x\big] + \ldots + \big[(-1)\, p_{n-2} x^{n-2}\big] + \big[(-1)^0 p_{n-1} x^{n-1}\big]\Big).$$

Multiplying ($x - \alpha$) into each square bracket term gives

$$p(x) \quad = \quad \left[(-1)^{n}\alpha p_{0} + (-1)^{n-1}p_{0}x\right] +$$

$$\left[(-1)^{n-1}\alpha p_{1}x + (-1)^{n-2}p_{1}x^{2}\right] + \dots +$$

$$\left[(-1)^{2}\alpha p_{n-2}x^{n-2} + (-1)\,p_{n-2}x^{n-1}\right]$$

$$+\left[(-1)^{1}\alpha p_{n-1}x^{n-1} + (-1)^{0}p_{n-1}x^{n}\right]$$

The terms with the same power of x can now be grouped together. The first term, $(-1)^n \alpha p_0$, and the last term, $(-1)^0 p_{n-1} x^n$, are the only stand-alone terms. So the polynomial of degree n having n positive roots can be rewritten as

$$p(x) \quad = \quad (-1)^{n}\alpha p_{0} + \sum_{j=1}^{n-1}\left[(-1)^{n-j}\left(\alpha p_{j} + p_{j-1}\right)x^{j}\right] + p_{n-1}x^{n}$$

In the summation block, $(- 1)^{n-j}$ is the part that decides the sign of the coefficient as p_j , α are positive terms.

Hence, the polynomial of degree n having n positive roots also has nonzero coefficients with alternating signs.

3) Conclusion: Since both the base case and the inductive step have been proven, by mathematical induction the proposition holds for every natural number n.

Corollary 3 *If a polynomial p of degree n has n negative roots, then its coefficients are all nonzero and agree in sign.*

Proof. We apply Proposition 2 to the polynomial $p(-x)$. Hence, $p(-x)$ has n positive roots and by Proposition 2, all its coefficients are nonzero and have alternating signs. So, all of the coefficients of $p(-(-x))= p(x)$ are nonzero and agree in sign.

Proposition 3 *If there is exactly one sign change in the coefficients, then p has at least one positive root.*

Proof. By Result 1 the sign of p_0 determines the sign of $p(x)$ for sufficiently small x and by Result 2 the sign of p_n determines the sign of $p(x)$ for sufficiently large x. Since the coefficients have exactly one sign change, p_0 and p_n disagree in sign. Therefore $p(x)$ has a different sign near 0 and infinity. By continuity, $p(x) = 0$ for some $x > 0$.

Using an argument from the days before calculus was invented, we now show that there is exactly one positive root if there is exactly one coefficient sign change.

We start with a simple observation on $\sum_{j=0}^{k-1} x^j$:

Observation 1
Let $\varphi_0(x) = 0$, $\varphi_k(x) = \varphi_{k-1}(x) + x^{k-1}$ *for* $k = 1, \ldots,$ *and* $x>0$.

Then $\varphi_k(x)$ *satisfies a relation:*

$$\varphi_m(x) > \varphi_k(x) \text{ for } m > k \text{ when } x > 0.$$

Proposition 4 *If there is exactly one sign change in the coefficients, then p has exactly one positive root.*

Proof. By Proposition 3, there is at least one positive root. Let $\alpha > 0$ be the smallest and form the polynomial $f(x) = p(\alpha x)$. Note that f has degree n. The coefficients of this new polynomial have the same signs as the original and the smallest positive root is shifted to $x = 1$.

Clearly, *f* also has exactly one coefficient sign change. Let's say that the last k terms of *f* have non-negative signs (some of these terms could be zero), and the first $n - k + 1$ terms have a non-positive sign (some of these terms could be zero).

We now show that *f* is non-zero when x is not equal to 1 and hence that $x = 1$ is a simple root for *f*. In doing so, we show that α is a simple root of p.

Splitting the positive and negative terms out, we write $f(x) = q(x) - r(x)$, where the polynomials q and r have nonnegative coefficients. Factoring, we have (remembering that $f(1) = 0$)

$$\begin{aligned} f(x) &= f(x) - f(1) \\ &= \left\{ \begin{array}{l} \left[q_{n-k+1}x^{n-k+1} + q_{n-k+2}x^{n-k+2} + \ldots + q_{n-1}x^{n-1} + q_n x^n\right] \\ \qquad - \left[r_0 + r_1 x + \ldots + r_{n-k-1}x^{n-k-1} + r_{n-k}x^{n-k}\right] \end{array} \right\} \\ &\quad - \left\{ \begin{array}{l} \left[q_{n-k+1} + q_{n-k+2} + \ldots + q_{n-1} + q_n\right] \\ \qquad - \left[r_0 + r_1 + \ldots + r_{n-k-1} + r_{n-k}\right] \end{array} \right\} \end{aligned}$$

and after regrouping terms, we get (for $x \neq 1$)

$$\begin{aligned} f(x) &= \sum_{j=n-k+1}^{n} q_j (x^j - 1) - \sum_{j=0}^{n-k} r_j (x^j - 1) \\ &= (x-1)\left[\sum_{j=n-k+1}^{n} q_j \frac{(x^j - 1)}{(x-1)} - \sum_{j=0}^{n-k} r_j \frac{(x^j - 1)}{(x-1)}\right]. \end{aligned}$$

By Observation 1,

$$\phi_j(x) = 1 + x + x^2 + \ldots + x^{j-2} + x^{j-1}$$

and so

$$(x-1)\left[\phi_j(x)\right] = \left[x + x^2 \ldots + x^{j-1} + x^j\right] + \left[-1 - x \ldots - x^{j-2} - x^{j-1}\right]$$
$$= x^j - 1$$

and hence we have shown that

$$f(x) = (x-1)\left[\sum_{j=n-k+1}^{n} q_j \phi_j(x) - \sum_{j=0}^{n-k} r_j \phi_j(x)\right]$$

$$= (x-1)\, s(x)\ .$$

We now show that $s(x)$ is positive for all $x > 0$, and thus $x = 1$ is a simple root of $f(x)$.

Given that $n \geq j \geq n - k + 1$, by Observation 1,

$$\Phi_j(x) \geq \Phi_{n-k+1}(x) > 0.$$

Hence,

$$\left[\sum_{j=n-k+1}^{n} q_j \phi_j(x)\right] \geq \left[\sum_{j=n-k+1}^{n} q_j \phi_{n-k+1}(x)\right] \quad \ldots(i)$$

We do not have a strict inequality above as j could equal $n - k + 1$ or k maybe 1! In these cases, the expressions on either side equal each other.

Given that $0 \leq j \leq n - k$, by Observation 1,

$$\phi_j(x) < \phi_{n-k+1}(x)\ .$$

Hence,

$$-\left[\sum_{j=0}^{n-k} r_j \phi_j(x)\right] > -\left[\sum_{j=0}^{n-k} r_j \phi_{n-k+1}(x)\right] \quad \ldots(ii)$$

Adding (i) and (ii), we get for any $x > 0$,

$$s(x) > \left\{\phi_{n-k+1}(x)\left[\sum_{j=n-k+1}^{n} q_j - \sum_{j=0}^{n-k} r_j\right]\right\}$$

$$= \phi_{n-k+1}(x) f(1)$$

$$= 0$$

We conclude therefore that *f(x)* and hence *p(x)* has exactly one positive root.

2.4. The Proof of Descartes' Rule of Signs

Lemma 2 *If a polynomial q exhibits m coefficient sign changes, then for any* $\alpha > 0$, *the polynomial* $p(x) = (x-\alpha)q(x)$ *exhibits at least* $m + 1$ *sign changes.*

Proof. Let the degree of *q(x)* be *n*. *q(x)* can be represented as

$$\sum_{j=0}^{n} q_j x^j .$$

Then, $p(x) = (x-\alpha)q(x)$ can be written as

$$p(x) = \left[q_0 x - \alpha q_0\right] + \left[q_1 x^2 - \alpha q_1 x\right] + \dots$$
$$+ \left[q_{n-1} x^n - \alpha q_{n-1} x^{n-1}\right] + \left[q_n x^{n+1} - \alpha q_n x^n\right].$$

After regrouping terms based on powers of x,

$$p(x) = -\alpha q_0 + \left[q_0 x - \alpha q_1 x\right] + \dots$$
$$+ \left[q_{n-1} x^n - \alpha q_n x^n\right] + q_n x^{n+1}$$

$$= -\alpha q_0 + \sum_{j=1}^{n} \left(q_{j-1} - \alpha q_j \right) x^j + q_n x^{n+1}.$$

This says that $p_{n+1} = q_n$ and hence both have the same sign.

We will call q_k a sign-change coefficient if

i) q_k is non-zero,
ii) there exists $k^* > k$ such that q_{k^*} has the opposite sign to q_k and
iii) $q_j = 0$ for $k + 1 \leq j \leq k^* - 1$.

Let q_m (necessarily, $m < n$) be the first sign-change coefficient as we come down from q_n. From the formula $p_{m+1} = q_m - \alpha q_{m+1}$ we obtain that p_{m+1} and q_m agree in sign. Since p_{n+1} and q_n have the same sign, and q_m and p_{n+1} have opposite signs, we get that there is at least one sign change in the coefficients p_j as j ranges from $m + 1$ to $n + 1$.

The same reasoning can be applied to any two successive sign change coefficients q_k and q_m to conclude that p_{k+1} and p_{m+1} have the same signs respectively, and hence there must be at least one sign change in the coefficients p_j as j ranges from k to m. Thus there are at least as many sign changes in p as there are in q.

Let q_δ denote the last sign-change coefficient. Clearly $\delta \geq 0$. Without loss of generality let us assume $q_\delta > 0$ (a similar argument works if the sign of q_δ is reversed). This means $p_{\delta+1} = q_\delta - \alpha\, q_{\delta+1} > 0$ and all q_j for $0 \leq j \leq \delta - 1$ are non-negative. We get that $p_0 = -\alpha q_0 < 0$ (recall the standing assumption that the zeroeth coefficient is non-zero). Hence there is one more sign change from $p_{\delta+1}$ to p_0.

Putting all the arguments above together, there is at least one more sign change in coefficients of p than those of q.

Corollary 4 *A polynomial p with k positive roots has at least k coefficient sign changes.*

Proof. We proceed by induction on k.

1) Base case: For $k = 1$, the result is immediate since if the polynomial has zero coefficient sign changes, i.e., if coefficients are both nonnegative or nonpositive, then it cannot have a positive zero. Note that at least one of them (the top coefficient) has to be non-zero by our assumption at the beginning.

2) Inductive step: We assume that any polynomial with $k - 1$ positive roots has at least $k - 1$ sign changes and show that a polynomial p with k positive roots has at least k sign changes.

   We write $p(x)=(x - \alpha)q(x)$, where $\alpha > 0$ is a root of p. Since we know that p has k positive roots, q is a polynomial with $k - 1$ positive roots and by inductive step 2) it has at least $k - 1$ sign changes. By Lemma 2, p has at least k sign changes.

3) Conclusion: Since both the base case and the inductive step have been proven, by mathematical induction the statement $P(k)$ holds for every natural number k.

Corollary 5 *A polynomial with k positive roots has more than k nonzero coefficients.*

Proof. By Corollary 4, a polynomial with k positive roots has at least k sign changes, hence at least $k + 1$ coefficients for the sign changes to occur between.

Corollary 4 hence proves the first part of Descartes' Rule of Signs.

Theorem 1 [Descartes' Rule of Signs—I] *The number of positive roots of a polynomial p with real coefficients does not exceed the number of sign changes of its coefficients.*

To show further that the difference between the number of sign changes and the numbers of roots is even, we employ a cute observation on parity:

Proposition 5 [Parity] *The parity, i.e. the remainder upon division by 2, of the number of sign changes in a sequence of nonzero real numbers s_j, $j = 0, \ldots, n$ is equal to the number of sign changes in the two element subsequence $s_0 s_n$.*

Proof. Let σ_j be the sign of s_j. Then the ratio σ_j / σ_{j+1} is -1 at each sign change and 1 otherwise.

Therefore,

$$(-1)^{\#\ \textit{sign changes}} = \prod_{j=0}^{n-1} \left(\frac{\sigma_j}{\sigma_{j+1}} \right).$$

But since

$$\prod_{j=0}^{n-1} \left(\frac{\sigma_j}{\sigma_{j+1}} \right) = \frac{\sigma_0}{\sigma_1} \times \frac{\sigma_1}{\sigma_2} \ldots \times \frac{\sigma_{n-2}}{\sigma_{n-1}} \times \frac{\sigma_{n-1}}{\sigma_n} = \frac{\sigma_0}{\sigma_n},$$

$$(-1)^{\#\ \textit{sign changes}} = \frac{\sigma_0}{\sigma_n}.$$

This says that the difference between the number of sign changes in the whole sequence and the number of sign changes (i.e. 0 or 1) in the subsequence $s_0 s_n$ is an even nonnegative integer. Or, that the remainder upon division by 2, of the number of sign changes in the sequence of nonzero real numbers $\{s_j\}$, is equal to the number of sign changes in the two element subsequence $s_0 s_n$.

Lemma 3 *If a polynomial q with real coefficients exhibits m sign changes, then for any* $\alpha > 0$*, the polynomial* $p(x)=(x-\alpha)q(x)$ *exhibits* $m+1+2l$ *sign changes for some integer* $l \geq 0$.

Proof. By Lemma 2, p has at least $m + 1$ sign changes.

We apply proposition 5 to the coefficients of $p(x)$. Only all the nonzero coefficients of p are put into the sequence $\{s_j\}$. Then, proposition 5 can be applied to this sequence of coefficients. The signs of s_0 and s_n are the signs of the trailing and leading coefficients respectively. The signs of the trailing and leading coefficients don't change. Proposition 5 says that the sign changes between the first and last terms must be dropped in pairs given that the signs of the trailing and leading coefficients are constant. Hence, p has $m + 1 + 2l$ sign changes for some integer $l \geq 0$.

Theorem 2 [Descartes' Rule of Signs—II] *The number of positive roots of a polynomial p with real coefficients does not exceed the number of sign changes of its coefficients and differs from it by a nonnegative multiple of two.*

Proof. By Corollary 4, the number of positive roots in p doesn't exceed the number of sign changes in its coefficients. And, by Lemma 3, the difference between the number of sign changes in p and the number of positive roots it has is a nonnegative multiple of two.

2.5. Working Backward

Given a polynomial p, we have seen how to deduce

i) The number of sign changes, S, of p
ii) The sign sequence $\{s_j\}$ of the coefficients of p, from which i) can be determined
iii) The possible number of positive roots (of p) that can be determined using the rule of signs when either S or $\{s_j\}$ is given.

Note that s_j is zero when the corresponding coefficient in p is zero.

In working backward, we are given a sign sequence $\{s_j\}$ with n elements from which we can calculate S as in i). We are then asked to find a polynomial whose coefficients have signs matching $\{s_j\}$ (in particular the polynomial is of degree n) and the number of positive zeros of the polynomial is any element in the set

之 = $\{S, S-2, S-4, ..., S_0\}$, where $S_0 = 1$ when S is odd and $S_0 = 0$ when S is even.

Note that 之 is the set of all possible counts of positive zeros of a polynomial with S sign changes allowed by the Rule of Signs.

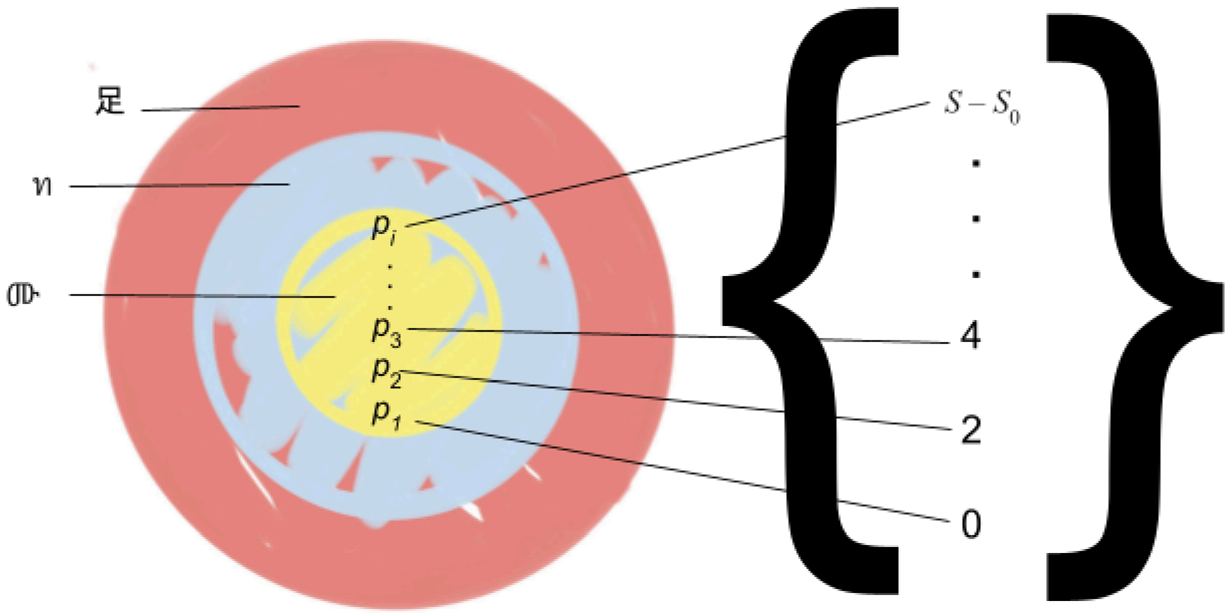
Grabiner [2] proves that this can indeed be done.

Let's illustrate Grabiner's [2] conclusion better. Let the class of all polynomials be denoted by足. We'll define the map F from足to the set of nonnegative even integers as

$$F(p) = S - Z_{positive},$$

where $Z_{positive}$ is any element from 之.

When given some $\{s_j\}$ with n elements, we can calculate S as in i). In the diagram below, there exist subsets ท and ထ of 足, where ท consists of all polynomials of degree n and ထ all polynomials of degree n having a coefficient signs matching the sequence $\{s_j\}$. Within ထ, different choices of the magnitudes of the coefficients will produce different p's having differing number of positive roots. Grabiner [2] constructs for each element in 之 a polynomial in ထ having that many number of positive roots. The curly brackets indicate the range of $F(p_i)$; p_i is a polynomial with $S - S_0$ positive roots.



From this work by Grabiner [2], it follows that the range of F equals all the nonnegative even numbers! This neatly shows that Descartes' Rule of Signs cannot be further narrowed down or refined.

References

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[2] Grabiner, D. J., 1999, Descartes' Rule of Signs: Another construction, The American Mathematical Monthly, Vol 106(9), 1999, p. 854-856.

[3] Levin, S. A., 2002, Descartes' Rule of Signs - How hard can it be?: http://sepwww.stanford.edu/data/media/public/oldsep/stew_save/descartes.pdf.

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Appendix

The code below calculates the possible number of positive, negative, and zero roots when given a polynomial with its coefficients ordered by powers of *x*.

```
#returns the number of sign changes in a given polynomial

def SignChanges(coef = [], *args):
 S=0
 for i in range(len(coef)-1):
   check = (coef[i])*(coef[i+1])
   if check < 0:
     S+=1
   else:
     pass
 return S

#prints number of possible positive, negative, and zero roots

def NumberOfZeroes(Sp,Sn,p):
 print("The possible number of positive roots are:", end = " ")
 l = int(Sp)
 while (l != 0) and (l != 1):
   print(l,",", end = "  ")
   l -= 2
 print(l)
 print("The possible number of negative roots are:", end = " ")
 m = int(Sn)
 while (m != 0) and (m != 1):
   print(m,",", end = "  ")
   m -= 2
 print(m)
 Z = 0
 print("The number of zero roots are:", end = " ")
```

```
 for i in p:
   if i == 0:
     Z += 1
   else:
     break
 print(Z)

p = []
q = []
n = int(input("enter the degree of the polynomial: "))

for x in range (n+1):
 a = float(input("what's the coefficient of x^%d? " %(x)))
 p.append(a)
 if x%2 == 0:
   q.append(a)
 else:
   q.append(-a)

print(NumberOfZeroes(SignChanges(p) , SignChanges(q) , p))
```